Why 25+4 might be 54
Children's Interpretations of Uncompleted Equations

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This article stresses the importance of flexible interpretations of equations and the meaning of the equality sign from primary onwards. The common comprehension of the equality sign as a call to compute has to be complemented by a relational sight, which focuses on the equivalence of terms on both sides. Either interpretation might be appropriate in different situation. The daily primary education is found as being conductive to the computing sight -whereas the relational sight has to be supported more explicitly. Some results of a research project indicate possible shifts in understanding, awareness, and interpretation of equations by young children.

Theoretical Framework

In primary school mathematics children should learn to handle basic operations. They are used to solve tasks by finding a concrete result. Beside this important branch of mathematical education, numbers should not be regarded as magnitudes to add to or subtract from each other but as objects in relations defined by operations.

To regard equations as such (cf. DAVIS, 1997) one could not trust on spontaneous development but has to challenge the children's thoughts by suitable tasks. Younger children tend to handle numbers as magnitudes but not as relations (cf. STERN, 1998). WINTER already claims in 1982 to support argumentations of more 'algebraic quality' in primary school mathematics. He analysed the existing school practice and states that computing -without any further thoughts- is often favoured by both children and teachers.

The equivalence of terms could be based on the equivalence of value but also on e.g. symmetry, analogy, or permanence principles, which are claimed algebraic strategies (cf. FREUDENTHAL, 1983). The algebraic strategies focus on relations between numbers beside the value.

For pre-algebra or algebraic thinking is not implemented in the German primary syllabus, these strategies sound foreign to primary teachers and are usually not taught. Although some publications and researches in the arithmetical field stress that arithmetic is essentially more then learning to calculate, the common sense of contents of primary mathematics does not differ that much from the "Rechenunterricht" as it was in the 60s.

The modern teaching methods (learning by discovery, regarding the child as the active and responsible subject of its own learning process, etc.) are increasingly accepted, but the mathematical content is not questioned.

Although higher order thinking skills (describe, conjecture, and reason) are aimed for, these skills often refer to 'solutions'. This article stresses the difference between the description of a correct computation (and the discussion of individual solving strategies) and the meta-level description of an equation (and the relation of the numbers given in there). The latter relate to the development of a conceptual sight whether the other ones refer to an empirical sight. To see the general in a given example (cf. MASON in KAPUT) is an important ability to get into grips with what the mathematical object might be.

Recent international studies verified the hypothesis that projects that support arithmetical abilities have an impact on algebraic competencies (cf. LIVNEH/LINCHEVSKI, 2003). Therefore an intervention in the arithmetical area may be fruitful for future learning of algebra.

Methods and Sample

The research wants to inquire the spontaneous reactions to and interpretations of equations, who offer the opportunity to take up a computing or a relational position. Tied in with the found status quo a project is started to facilitate developments in interpretation. These developments are analysed with various means.

Within the project substantial learning environments (SLE) have been developed, which emanate from typical arithmetical problems (cf. WITTMANN, 2002). The chosen problems offer both the chance to work on by computing and to solve the task by using the relations between the given numbers and operations without 'getting the result'. These different interpretations of the task are labelled as the *computing sight* on the one hand and the *relational sight* on the other hand.

In order to fathom the daily opportunities and to become aware of unused scopes in mathematics lessons, a school project is launched. In co-operation with six different primary school teachers the learning environments were evaluated over a period of up to 9 month in six classes. Three classes start the project in the middle of year 2, three in the middle of year 3. The projects therefore end for the younger ones in the middle of year 3 and for the older ones in the middle of year 4. The age of the children involved range from 8 years in the beginning up to 10 years at the end of the project. In total 147 children participated in the project.

In the beginning of the project the previous knowledge is ascertained by a pre-test. At the end of the project the same test is given as a post-test to investigate any variation in solving strategies and results. The test contains both closed and open questions. Closed questions need to be answered by fitting in a missing number in an equation, open questions ask for solving strategies and inner thoughts.

The SLE tasks are presented in workbooks, which are worked on individually by each child in the project classes. If the teacher wants the children to work in pairs or groups, the workbooks should be completed individually as well. In order to fix the different methods the children are asked to tick boxes below each task whether they found the answer on their own, in pairs, or in groups.

The project lessons are not methodically defined. The co-operation teachers should work in the given SLEs into their daily mathematics lessons. The SLEs are proposals commented in a teacher's guidebook where possible solutions by children and arrangements are provided. The teachers experience and the influence of the different teaching styles are evaluated by an accompanying video-study.

The individual competences of all 147 children are ascertained by the workbook solutions and the results in the pre- and post-test. In order to get more insight into possible learning biographies, two children of each class are interviewed four times during the project. The first and last interviews are held at the same time as the pre- and post-test, the second and third interview are carried out during the project time. The participating children are not chosen randomly but are identified by the reaction to one certain task of the pre-test. One 'computing' and one 'relational' child are taken out of each class.

Within this paper only one task can be dealt with in more depth to exemplify the children's interpretations. Afterwards one case study sheds light on a typical learning development dealing with this particular task during the whole project time.

Design of an Exemplary Task - 25+4=__+25

During the project the children are asked to complete 25+4=__+25 in the pre-test, the post-test, and in the workbooks presented in the project-lessons (for typical interpretation of turnaround tasks also cf. WARREN, 2001). Likewise the task is one of the 'stable' items used in the interviews in order to record shifts in interpretation.

In the tests this task is one of the closed questions, which is solved by fitting in a number without any comments.

25+4=__+29
25+4=27+__
25+4=26+__
25+4=__+25

fig. 1

The equation also appeared in the workbook - actually it is the first task offered here. The design of the task slightly differs here. The equation is embedded into a column of tasks (fig. 1).

The column follows a certain pattern -which is 'disordered' by the missing second task (25+4=28+1). The missing task should prevent strategies, which fill in the ordered natural numbers without any further thoughts (cf. STEINWEG, 2001).

Within the workbooks the children are additionally asked to answer at what task they started working on the column and if they want to comment on some distinctive feature.

The one on one interview situation invites the children to think aloud. The video-taped comments are transcribed afterwards. Within
the first interview the child's interpretation is accepted as given. Thenceforth the second interview all children are confronted with the opposed kind of answers, in order to evoke a cognitive conflict, and to inspirit the dialogue.

First Results

The answers given to the exemplary task are categorised concerning the assumed interpretation of the equality sign. The results are shown below (fig. 2):

in %	Pre-test	Post-test	Workbooks
computing sight (29)	**65.2**	9.0	2.8
relational sight (4)	25.2	**89.5**	**89.5**
other solutions	3.7	1.5	7.7
not worked on	5.9	0.0	0.0

fig. 2: given answers to 25+4=__+25

A first sight diagnosis leaves the impression that the bigger part of the children joined the project with a computing sight. The equality sign is regarded as a call for computing (cf. WARREN, 2001). The next part of the right term is neglected. Some of the children categorised 'computing sight' also compute the 'new' right term 29+25 and add another equality sign and the solution 54 to the equation. This interpretation might be inspired by daily experiences of orally given task in a kind of 'chain' (start with 6, add 20, take away 12, etc.), which is quite popular in German primary schools.

The results in the post-test approve a shift of interpretation. Nearly 90% of the children 'see' the commutative task on both sides of the equality sign and use this relation between the addends to solve the task.

Within the workbooks nearly 90% of the participants fill in the fitting number 4, even though the task is given shortly after the pre-test. The disordered pattern misleads only one child, who fills in 0, 1, 2, 3. The pattern therefore supports the mathematically correct answer.

The children are asked to note whether they worked individually or not on this task. A quarter of the children state that they worked in pairs, 60% worked individually. The others do not take any notes.

The analysis of the notes concerning the task they started working on within the column give the following results: 77.6% started with the first task, 2.1% with the second, 0.7%

with the third, and 4.9% with the fourth task. The typical and common reaction to a column is to start with the first task given. The comments may not mirror the real action but yet they give information about the thinking process, because the note reflects the answer thought over. Seven children state to have started with the fourth task. This manner can be taken as an indicator that they have spotted the commutative task here and used it for the solution of the other tasks.

The children's notes about some features, which strike them, show, that most of the children comment on either the phenomenology of the column, i.e. the repeated left term (23.8%), or on the constant result (28.0%). 16.1% of the children describe the increase and decrease of the addends. The first sight reaction often sticks to 'visible' but static phenomena whereas some children already highlight the dynamical relations. The solution 4 does not indicate that the children's thinking is grounded on the relations. These additional notes rather show that most of the descriptions refer to computing arguments. However the equality sign is not regarded as a computing sign but as a relational sign by these children.

The results of the interview study pave the way to a deeper analysis of the thinking skills and strategies. The interviews held at different times throughout the project allow the analysis of changes concerning working on the task and awareness of what is the mathematical object.

A Case Study - Felix' Interpretations

At the beginning of the project Felix -a 2nd-Grader- is 8 years and 2 month old. Within the pre-test he leaves out the task, in the post-test he fills in the number 4. During the project he is interviewed several times.

In the first dialogue his answers corroborate the hypothesis that he regards the equality sign as a call to compute (line 3):

1 F then here the number that fits in here? *(tips on the empty line)**
2 I mhm
3 F simply twenty-five plus four *(points at the left term)* twenty-nine *(fills in 29)*

In the second and third interview Felix first reaction to the task is to fill in 29. The interviewer therefore (see design described above) confronts him with the solution of another child who filled in 4.

During the second interview Felix remarks that this child has to be older than he. This is taken as an indicator that he does not doubt the correctness of this solution but he can not understand the way of thinking.

4 I what do you notice, when you compare this to yours?
5 F that the number is smaller.
6 I mhm.
7 F considerably [emphatic] smaller.
8 I why has the child put in four then?
9 F mh. because there is written plus four.

In line 7 Felix makes clear that he sees the major differences in solutions. He links the answers to the given addend in the right term (line 9) but can not explain the relation.

His increased awareness of the different strategies to work on the equation can be shown in the third interview. Felix changes his mind after being confronted with the opposed solution.

10 F oh yes *(puts his hand at his head)* right! that is true [emphatic]! *(deletes the noted 29 and substitutes it by 4)*
11 I what have you recalled?
12 F mhm. [5 sec] that ehm *(points with the pencil at the left term)* that *(points at the right term)* must be equal on both sides (points again at the left term) and therefore there has to be 4 there *(points at the empty line).*

Felix interpretation shifts (line 10) not because of blind trust on the correctness of the other solution but he can explain the different strategy in his very own words (line 12). He either knows by now that the value of the terms on both sides has to be the same or that the same numbers has to appear on both sides.

In the final interview -Felix is now 8 years and 9 month old- he explicitly refers to the equivalence in value without any extrinsic motivation.

13	I	how do one find the fitting number?
14	F	because one knows that that ehm there when one there are *(points with the pencil on the left term)* 25 and 4 and then there has to be a 4 *(points at the empty line)* that are 29 here then *(points at the left term)* and then that are also 29 *(points at the right term)*

Felix interpretation of the equation differs from the one in the beginning. He is now paying attention to every number and operation appearing. He is no longer neglecting the second addend of the right term. Felix still computes but he relates to the equivalence constituted by the equality sign. Therefore his sight can be labelled as 'relational'.

Remarks

In this paper only one task of the various items used in the project is presented. Also Felix learning biography is of course only one out of 147. His argumentation undergoes a typical change though. Even the children, who always completed the task by filling in 4, are quite unsure whether this is the only sound interpretation of the task. In other interviews they comment on the solution 25+4=29+25=54 by "one can do so also" (Johanna), or "might be right" (Vanessa). Only one of the interviewed children dares to say "I won't compute it like this" (Kerstin). Therefore an interpretation from a relational sight has to be specifically supported from the very first start of schooling.

One of the major difficulties in secondary algebra is the poor ability to structure an equation (cf. MALLE, 1983). If an equation never appears as such in the primary education, the shift of interpretation is even harder for the students. Furthermore both secondary and primary teachers often believe that equations are dealt with like this, and that the children only have to swap numbers by letters. The major differences in the interpretation are underestimated. Variables -even as 'letters evaluated' (cf. KÜCHEMANN, 1978)- and the exposure to them are not provided by a basis, if the experiences in primary are limited to the computing sight (cf. KAPUT et al.). The presented study has shown that shifts (cf. YACKEL, 1997) from an empirical-computing sight, which sticks to the given numbers and the equality sign as call to compute, to a relational sight, as a precursor of conceptional-algebraic thinking, can be nurtured.

The uttered thoughts indicate some mathematical activities, which do no longer belong to routine mathematics -for the children. This shows that even in elementary mathematics 'unspoken changes of perspectives' (cf. MALARA & IADEROSE, 1999) can be identified. These changes have to be made 'speakable' for the children and for the teachers. Awareness for the transitions, which are founded very early and are regarded by DOOREN, et al. (2001) as 'precursors for algebraic thinking' (p. 359), is vital. The shown examples put the finger on the important pivotal where the change of perspective can be successfully made. Both interpretations are fruitful working on an equation. Awareness of the differences and the suitability might be the only way to prevent the constitution of two different 'mathematics' -a computing arithmetic and an algebra dealing with relations- as one can find in the reactions of secondary school children (cf. CERULLI & MARIOTTI, 2001).

Basic skills and arithmetical routines are only one side of the coin. The awareness of objects dealt with on a meta-level is another important branch even for younger children. The hereby shown first attempts to design suitable, 'potentially algebraic' tasks are one possible way to enrich primary school mathematics. Additionally to the description of, conjecture, and reason about solving strategies, the description of the structure of equations, the conjecture about symmetries inside or analogies with other equations, and reason about general relations should become common activities. First steps towards this pre-algebraic thinking are to question whether a computation is needed for a solution at all.

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* The German interviews are translated by the author with the attempt to keep both the children's language and its original meaning. Interviewer (I) and Felix (F). The passages given are enumerated as they appear here. The child's actions are noted in italics in rounded brackets *(fills in 29)*. Important phonetic information is given in brackets [emphatic].